Roots of polynomials under differential flows

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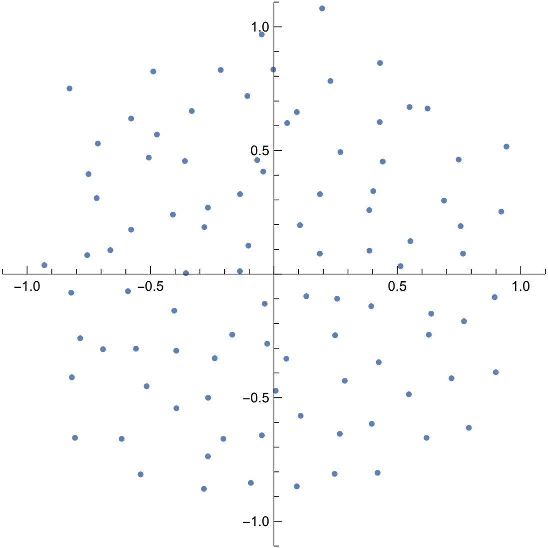
Basic question

- How do the roots of a polynomial change as we change the polynomial?
- Main examples in this talk: **heat flow** and **repeated differentiation**
- Will consider both operations in two cases: **real roots** and **complex roots**
- Will find a close connection to **random matrix theory** and **partial differential equations**

Differentiation example



POLYNOMIALS WITH ALL REAL ROOTS: HEAT FLOW

Heat flow on polynomials: definition

- If $p(z)$ is a polynomial on $\mathbb{C}$, define **heat operator**

$$\exp\left\{\frac{\tau}{2}\frac{d^2}{dz^2}\right\}p(z)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\tau}{2}\right)^n\frac{d^{2n}p}{dz^{2n}},\quad z\in\mathbb{C},$$

  as a terminating power series, all $\tau\in\mathbb{C}$
- If $\tau=t$ is real and positive and $z=x$ is real, the function

$$u(x,t):=\exp\left\{\frac{t}{2}\frac{d^2}{dx^2}\right\}p(x),\quad x\in\mathbb{R},$$

  satisfies the standard **heat equation**

$$\frac{\partial u}{\partial t}=\frac{1}{2}\frac{\partial^2 u}{\partial x^2}$$

Heat operator on polynomials: normalization

- If $p(z)$ is a polynomial of degree N, convenient to **normalize** heat operator with N in denominator:

$$\exp\left\{\frac{\tau}{2N}\frac{d^2}{dz^2}\right\}p(z)$$

- This normalization will make the evolution of zeros behave well when $N\rightarrow\infty$

Backward heat flow on polynomials

- Now take $\tau = -t$ and consider **backward heat operator**

$$\exp\left\{-\frac{t}{2N}\frac{d^2}{dz^2}\right\}, \quad t > 0,$$

on polynomials

Theorem (Pólya–Benz 1934)

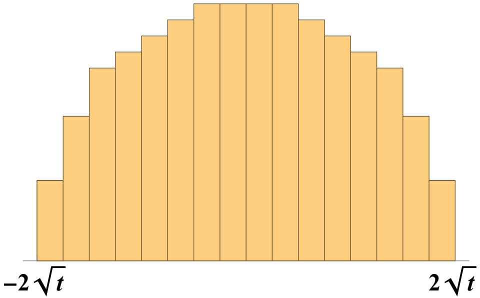
If p has all real roots, so does

$$\exp\left\{-\frac{t}{2N}\frac{d^2}{dz^2}\right\}p(z)$$

for all $t > 0$.

Backward heat operator: first example

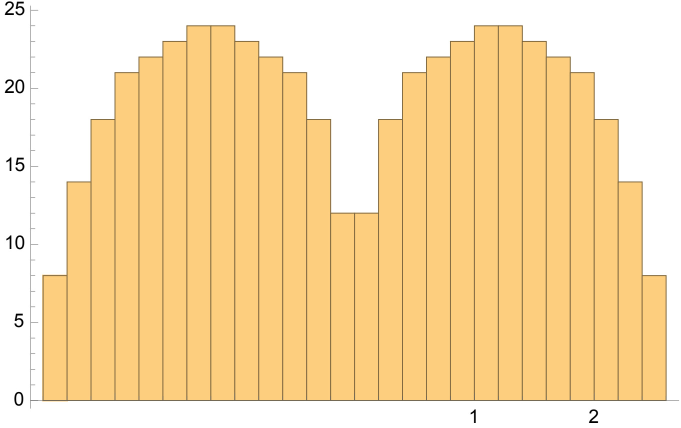
- Apply to z^N, get scaled **Hermite polynomial**
- Histogram of zeros of $e^{-\frac{t}{2N}\frac{d^2}{dz^2}}(z^N)$ with $N = 200$



- Zeros have asymptotically **semicircular shape** on $[-2\sqrt{t}, 2\sqrt{t}]$

Backward heat operator: second example

- Take $p(z) = (z-1)^{N/2}(z+1)^{N/2}$
- Half zeros at 1, half at -1
- Histogram of zeros of $e^{-\frac{t}{2N}\frac{d^2}{dz^2}}p$ with $N = 500$, $t = 1$



Connection to random matrix theory

- **GUE**: Gaussian unitary ensemble
- Take $N \times N$ Hermitian random matrix X with entries on and above diagonal independent
- Complex Gaussian with mean zero and variance $1/N$ off diagonal
- Real Gaussian with mean zero and variance $1/N$ on diagonal
- Eigenvalues asymptotically have **semicircular distribution** on $[-2, 2]$

Connection to random matrix theory

- Take sequence of real-rooted polynomials p^N of degree N
- Assume root distribution converges to prob. measure μ
- Make **diagonal matrix** X_0^N with roots of p^N on diagonal
- Take X^N to be GUE matrix

Claim

Roots of $e^{-\frac{t}{2N}\frac{d^2}{dz^2}}(p^N(z))$ resemble eigenvalues of $X_0^N + \sqrt{t}X^N$, which can be computed using **free convolution** *of μ with a semicircular distribution.*

- So: backward heat flow is like adding a GUE

Free convolution with semicircular distribution

- Define **Cauchy transform** of measure μ on $\mathbb{R}$ by

$$C_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x), \quad \operatorname{Im} z > 0.$$

- Holomorphic on upper half-plane
- Can recover μ from C_μ by Stieltjes inversion formula

$$d\mu(x) = -\frac{1}{2\pi} \lim_{\varepsilon \to 0^+} \left(\operatorname{Im} C_\mu(x + i\varepsilon)\ dx \right)$$

Free convolution with semicircular distribution

Theorem (Voiculescu/Kabluchko)

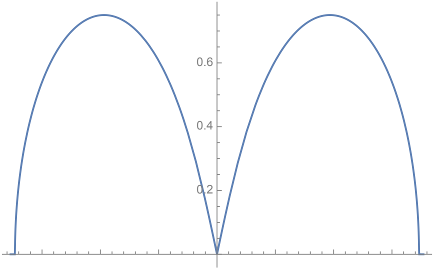
If polynomials p^N has real roots and the distribution of roots converges to μ, then the distribution of roots of $e^{-\frac{t}{2N}\frac{d^2}{dz^2}}p^N$ converges to a measure μ_t whose Cauchy transform satisfies

$$\frac{\partial C}{\partial t} = -C\frac{\partial C}{\partial z}, \quad \operatorname{Im} z > 0,$$

- Can then solve this PDE using the **method of characteristics**
- Gives semi-explicit way to compute μ_t
- μ_t is **free convolution** $\boxplus$ of μ with semicircular measure of variance t

Roots at ± 1

- Take μ to have mass $1/2$ at 1 and mass $1/2$ at -1
- Describe polynomial p with zeros at ± 1
- Compute $\mu \boxplus \mathrm{sc}_t$ at, say, $t = 1$



- This gives limiting distribution of zeros of $e^{-\frac{1}{2N}\frac{d^2}{dz^2}}p(z)$

PART 2

POLYNOMIALS WITH ALL REAL ROOTS:
REPEATED DIFFERENTIATION

Repeated differentiation of polynomials with real roots

- Start with polynomial P^N of degree N with real roots
- Then differentiate $\lfloor Nt \rfloor$ times, $0 \leq t < 1$
- Number of deriv. proportional to N
- Roots remain real!
- Assume root dist. of P^N converges to μ
- Try to find limiting root dist. μ_t of $\lfloor Nt \rfloor$-th derivative

Connection to random matrix theory

- Assume (at first) that $t = 1 - 1/k$ with $k \in \mathbb{N}$
- Then $\mu_t = \mu^{\boxplus k} := \mu \boxplus \cdots \boxplus \mu$, rescaled by a factor of $1 - t$
- $\mu^{\boxplus k}$ is like adding k indep.Hermitian matrices with e.v. distribution μ
- Then extend definition to arbitrary t (i.e., fractional k)
- "Fractional free convolution" of Shlyakhtenko and Tao

Connection to random matrix theory

Theorem (Hoskins–Kabluchko, '21; Arizmendi–Garza-Vargas–Perales, '23)

If polynomials P^N have limiting root distribution μ then $\lfloor Nt \rfloor$-th derivative of P^N has limiting root distribution equal to

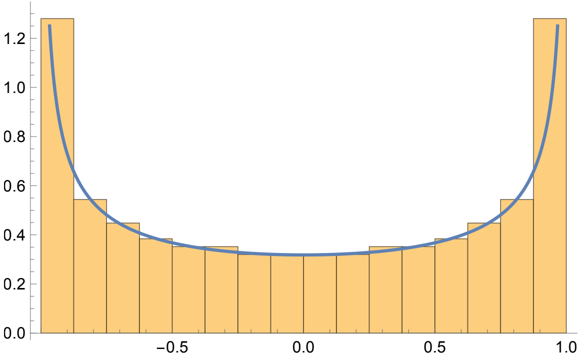
$$\mu^{\boxplus k}, \quad k = \frac{1}{1-t},$$

rescaled by a factor of $1-t$, for $0 \leq t < 1$.

- Results motivated by work of Steinerberger, 2019

Example: Roots at ± 1

- Take $p(z) = (z-1)^{N/2}(z+1)^{N/2}$; i.e. $\mu = \frac{1}{2}(\delta_1 + \delta_{-1})$
- Take $t = 1/2$—i.e., take $N/2$ derivatives—so $k = 2$
- Then $\mu^{\boxplus k} = \mu \boxplus \mu$ can be computed explicitly
- After rescaling, get "arcsin" distribution $d\mu_t(x) = \frac{1}{\pi}\frac{1}{\sqrt{1-x^2}}\,dx$



PDE for the Cauchy transform

- Use **rescaled** measure $(1-t)\mu_t$ of mass $1-t$
- Let $C(z,t)$ be Cauchy transform of $(1-t)\mu_t$
- Use PDE from Shlyakhtenko–Tao for Cauchy transform of $\mu^{\boxplus k}$

Theorem

The Cauchy transform $C(z,t)$ of $(1-t)\mu_t$ satisfies the PDE

$$\frac{\partial C}{\partial t} = \frac{1}{C}\frac{\partial C}{\partial z}.$$

- Compare to $\frac{\partial C}{\partial t} = -C\frac{\partial C}{\partial z}$ for backward heat flow

POLYNOMIALS WITH COMPLEX ROOTS: HEAT FLOW

Example: Characteristic polynomial of GUE

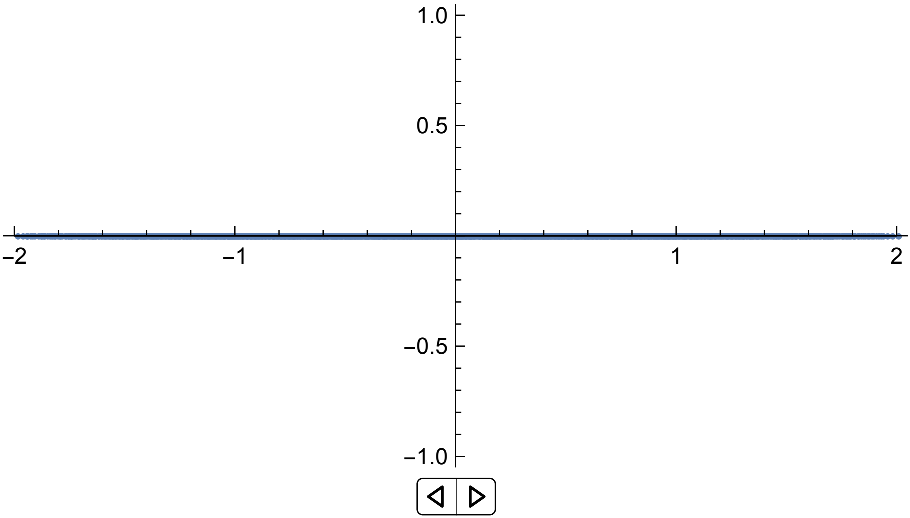
- Let p^N be char. poly. of GUE, zeros semicircular on $[-2, 2]$
- Applying *backward* heat operator gives semicircular dist. on *bigger* interval (width $4\sqrt{1+t}$)
- What about **forward** heat operator

$$\exp\left\{+\frac{t}{2N}\frac{d^2}{dz^2}\right\} p^N(z) \ ?$$

- Just change t to $-t$ (semicircular on *smaller* interval)?

Example: Characteristic polynomial of GUE

- Let's see!



Example: Characteristic polynomial of GUE

- Conjecturally, zeros $\to$ **uniform on ellipse** w/ semi-axes $2-t$ and t
- At $t=1$, zeros should become uniform on unit disk
- Heat flow changes "semicircular law" (s.c. on $[-2,2]$) to "circular law" (uniform on disk)!

Cauchy transform for measures in plane

- Compactly supported prob. measure μ with bounded density
- Define Cauchy transform as before:

$$C(z) = \int_{\mathbb{C}} \frac{1}{z-w}\, d\mu(w), \quad z \in \mathbb{C}$$

- But C will be **non-holomorphic** inside its support
- Ex: μ uniform on unit disk: $C(z) = \bar{z}$ in disk; $1/z$ outside
- Recover density of measure μ as

$$\frac{1}{\pi}\frac{\partial}{\partial \bar{z}} C(z)$$

General conjecture

Conjecture (Hall–Ho, 2022+)

Let μ_τ be limiting empirical measure of zeros of

$$\exp\left\{-\frac{\tau}{2N}\frac{d^2}{dz^2}\right\}p^N(z).$$

Then Cauchy transform $C(z,\tau)$ satisfies PDE

$$\frac{\partial C}{\partial \tau} = -C\frac{\partial C}{\partial z}. \tag{1}$$

- Here τ is *complex* variable; derivatives are Cauchy–Riemann ops.

$$\frac{\partial}{\partial \tau} = \frac{1}{2}\left(\frac{\partial}{\partial \tau_1} - i\frac{\partial}{\partial \tau_2}\right); \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

- **"Same" PDE** as in the real-rooted case!

Heuristic argument for conjecture

- Define Cauchy transform of zeros of polynomial

$$C^N(z,\tau) := \frac{1}{N}\sum_{j=1}^{N}\frac{1}{z - z_j(\tau)}$$

where $z_j(\tau)$ are zeros of heat-evolved polynomials

Theorem

The function C^N satisfies the PDE

$$\frac{\partial C^N}{\partial \tau} = -C^N\frac{\partial C^N}{\partial z} - \frac{1}{2N}\frac{\partial^2 C^N}{\partial z^2},$$

which **formally** *converges to the PDE in the conjecture as $N \to \infty$.*

- But not so easy to make a rigorous argument from this!

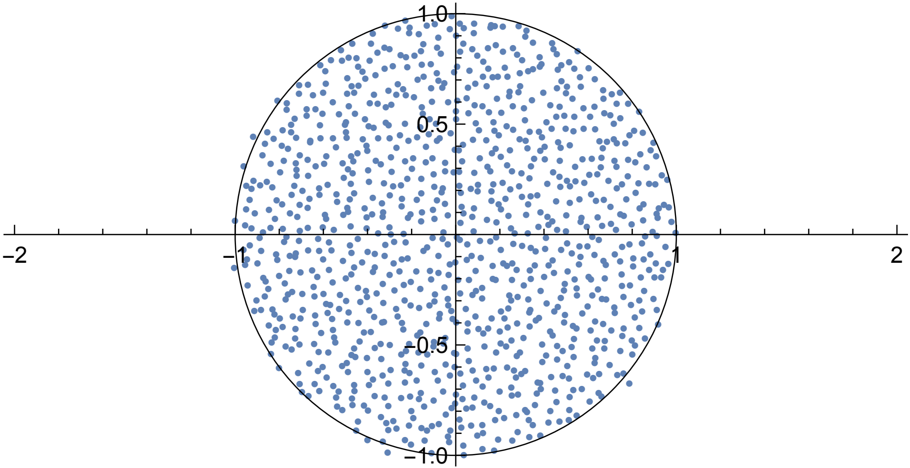
Example 1: Elliptic random matrix model

- Take X and Y be independent GUEs, $\tau \in \mathbb{C}$ with $|\tau| < 1$
- Take
$$Z = e^{i \arg(\tau)/2} \left(\sqrt{1 + |\tau|}\, X + i \sqrt{1 - |\tau|}\, Y \right)$$
- Eigenvalues uniform on ellipse with semi-axes $\sqrt{1 \pm |\tau|}$, rotated by $\arg(\tau)/2$
- $\tau = 0$ gives circular law
- **Theorem**: Log potential $S(z, \tau)$ of limiting e.v. distribution satisfies PDE in conjecture

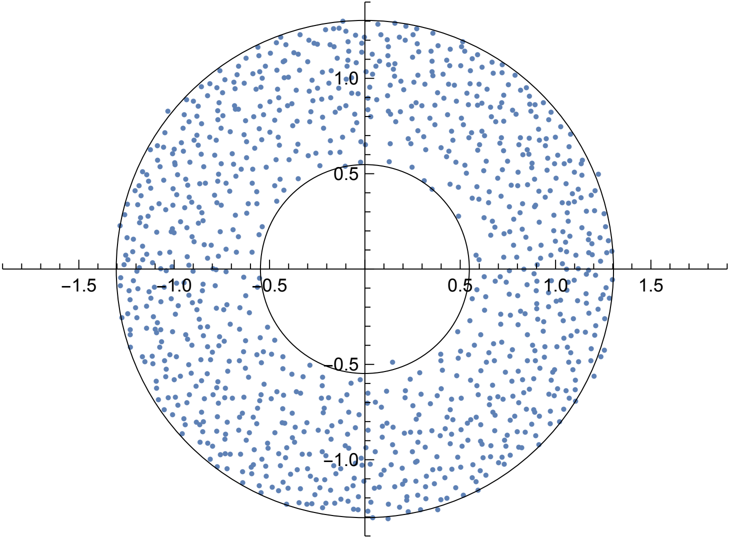
Example 1: Elliptic random matrix model

- Start with char. poly. of model with $\tau = 0$ (circular law)
- Then evolve by heat flow for time $t \in (0, 1)$
- **Conjecture says**: roots at time t should be uniform on ellipse with semi-axes $1 + t$ and $1 - t$
- And there are lots more similar examples from random matrix theory!

Example 1: Elliptic random matrix model



Example 2: Haar unitary plus elliptic



Rigorous results for random polynomials

- **Kabluchko–Zaporozhets**: large class of random polynomials with independent coefficients

$$p^N(z) = \sum_{j=0}^{N} \xi_j c_j^N z^j$$

- ξ_j: indep. and identically distributed random var.
- c_j^N are deterministic constants (with nice behavior as $N \to \infty$)
- Limiting distribution of zeros is **rotationally invariant** on a disk
- Essentially **any** rot. invariant measure on disk occurs for some c_j^N

Example: Weyl polynomials

- Take

$$W_N(z) = \sum_{j=0}^{N} \xi_j \frac{N^{j/2}}{\sqrt{j!}} z^j$$

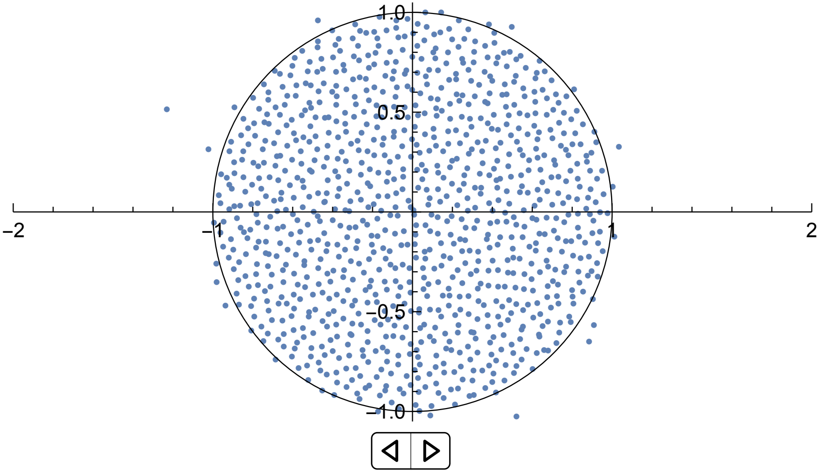
- Limiting distribution of zeros **uniform on unit disk**
- **Circular law** for random polynomials!

Rigorous results for random polynomials

Theorem (Hall–Ho–Jalowy–Kabluchko, 2023+)

The heat-evolved Kabluchko–Zaporozhets polynomials satisfy the Hall–Ho conjecture with probability one.
That is, the Cauchy transform of the limiting root distribution satisfies the claimed PDE, for sufficiently small τ.

Evolution of zeros of Weyl polynomials, $0 \leq \tau \leq 1$



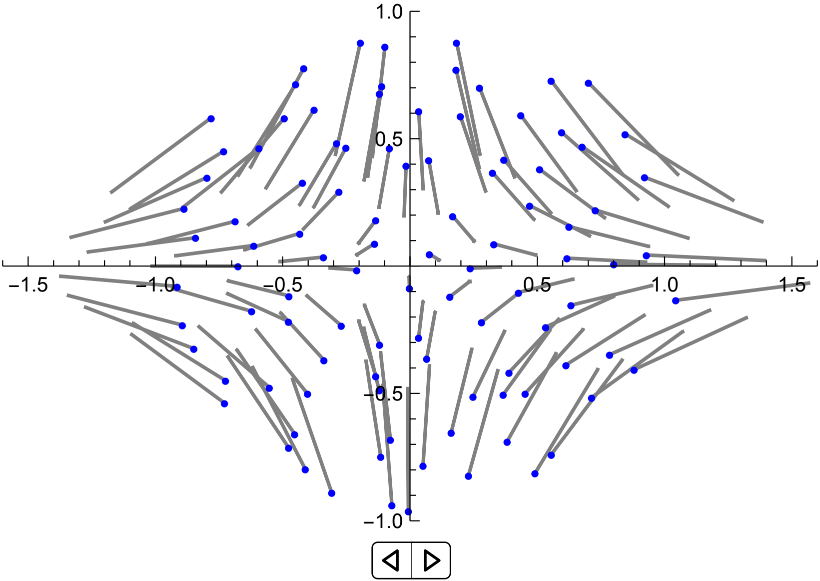
Transport behavior

- Expect zeros to evolve in **straight lines** with constant velocity
- Velocity given by the value of Cauchy transform at time 0
- These are **characteristic curves** of the relevant PDE

Theorem (Hall–Ho–Jalowy–Kabluchko, 2023+)

This behavior holds "at the bulk level." That is, for sufficiently small τ*, the measure* μ_τ *is the push-forward of* μ_0 *by map obtained by evolving along straight lines.*

Straight-line motion



POLYNOMIALS WITH COMPLEX ROOTS: REPEATED DIFFERENTIATION

PDE for Cauchy transform

- Expect Cauchy transform to satisfy
$$\frac{\partial C}{\partial t} = \frac{1}{C}\frac{\partial C}{\partial z} + \frac{1}{\bar{C}}\frac{\partial \bar{C}}{\partial z}.$$
away from points where $C = 0$
- Second term is present only because t is real
- "Essentially" same PDE as for real-rooted case
- Rigorous results for random polynomials

Transport behavior for random polynomials

- For random polynomials, we establish **transport behavior** based on the following idea.

Idea

Let μ_0 be the (radial) limiting root distribution of the initial polynomials and let $m_0(z)$ be its Cauchy transform. Then under repeated differentiation, roots evolve approximately **radially with constant speed** *according to*

$$z(t) \approx z_0 - \frac{t}{m_0(z_0)}$$

until they reach the origin, at which point they die.

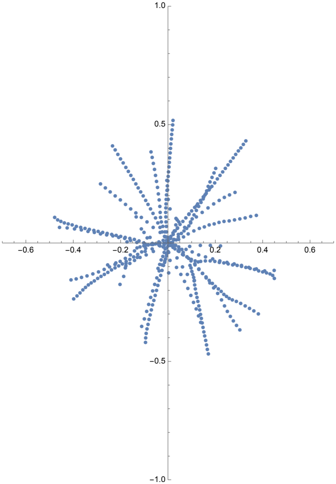
Transport behavior for random polynomials

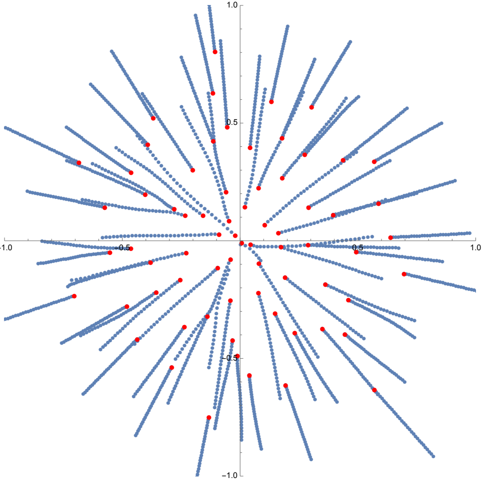
- We verify this idea rigorously at bulk level

Theorem (Hall–Ho–Jalowy–Kabluchko, 2023+)

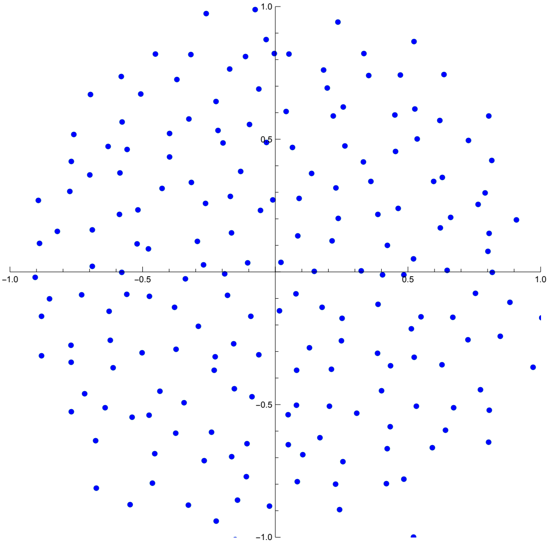
The limiting root distribution μ_t of $\lfloor Nt \rfloor$-th derivative is the push-forward of μ_0 under a transport map constructed according to the idea in the previous slide.

Transport behavior for random polynomials





Comparing to predicted straight-line motion



Examples from random matrix theory

- Look at limiting root distribution of $P^N(z^2)$
- I.e. take square roots (with both signs) of roots of $P^N(z)$
- Take bi-unitarily invariant ("radial") random matrix Z^N
- Match e.v.'s of Z^N to roots of P^N as $N \to \infty$

Theorem (Campbell–O'Rourke–Renfrew)

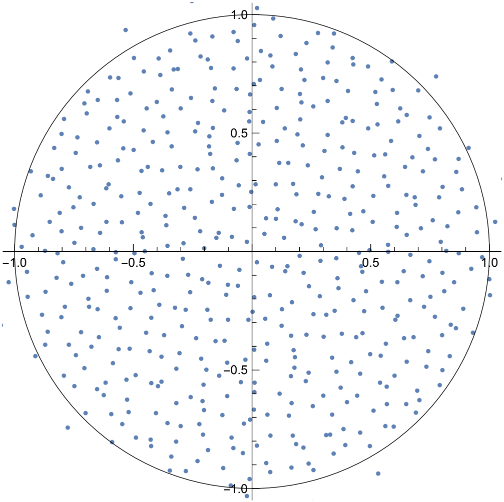
As $N \to \infty$, roots of $\lfloor Nt \rfloor$-th derivative of P^N, evaluated at z^2, match e.v.s of **truncation** *of Z^N to size $\lfloor N(1-t) \rfloor$*

- Equivalent to **fractional free convolution**

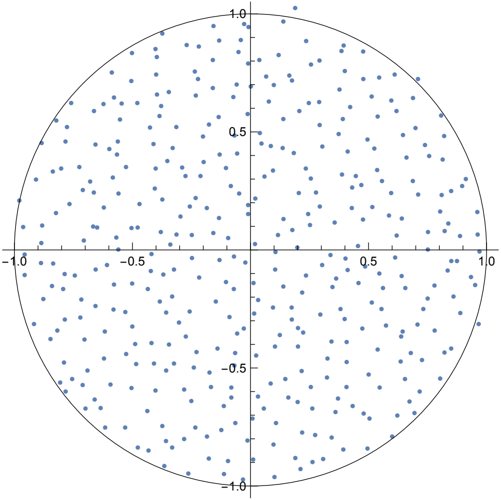
Examples from random matrix theory

- E.g. P^N are "exponential polynomials"; Z^N is Ginibre
- Initial roots/e.v.'s are uniform on unit disk
- At time t, both give uniform measure on disk of radius $\sqrt{1-t}$

Plot of roots of $P^N(z^2)$



Plot of truncated Ginibre matrix



Conclusion

THANK YOU FOR YOUR ATTENTION